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Gravitational and Yang-Mills instantons in holographic RG flows

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ABSTRACT: We study various holographic RG flow solutions involving warped asymptotically locally Euclidean (ALE) spaces of A_{N-1} type. A two-dimensional RG flow from a UV (2,0) CFT to a (4,0) CFT in the IR is found in the context of (1,0) six dimensional supergravity, interpolating between $AdS_3 \times S^3/\mathbb{Z}_N$ and $AdS_3 \times S^3$ geometries. We also find solutions involving non trivial gauge fields in the form of $SU(2)$ Yang-Mills instantons on ALE spaces. Both flows are of vev type, driven by a vacuum expectation value of a marginal operator. RG flows in four dimensional field theories are studied in the type IIB and type I′ context. In type IIB theory, the flow interpolates between $AdS_5 \times S^5/\mathbb{Z}_N$ and $AdS_5 \times S^5$ geometries. The field theory interpretation is that of an $N = 2$ $SU(n)^N$ quiver gauge theory flowing to $N = 4$ $SU(n)$ gauge theory. In type I′ theory the solution describes an RG flow from $N = 2$ quiver gauge theory with a product gauge group to $N = 2$ gauge theory in the IR, with gauge group $USp(n)$. The corresponding geometries are $AdS_5 \times S^5/(\mathbb{Z}_N \times \mathbb{Z}_2)$ and $AdS_5 \times S^5/\mathbb{Z}_2$, respectively. We also explore more general RG flows, in which both the UV and IR CFTs are $N = 2$ quiver gauge theories and the corresponding geometries are $AdS_5 \times S^5/(\mathbb{Z}_N \times \mathbb{Z}_2)$ and $AdS_5 \times S^5/(\mathbb{Z}_M \times \mathbb{Z}_2)$. Finally, we discuss the matching between the geometric and field theoretic pictures of the flows.

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1. Introduction

Holographic RG flow between D-dimensional (D=2,4) CFT's is one of the most studied aspects of the AdS/CFT correspondence [1]. The issue has been addressed both in the framework of D+1 (possibly gauged) supergravity [2, 3, 4, 5, 6, 7], and in the full 10-dimensional supergravity, where the internal space is typically a warped non-compact Calabi-Yau manifold [8, 9].

In this paper, we study RG flows in both two- and four- dimensional contexts. In the previous paper [10], we have studied RG flows in minimal six dimensional supergravity with Yang-Mills instantons turned on on $\mathbb{R}^4$. Here we still work in the framework of (1,0) six dimensional supergravity, but now we replace the transverse $\mathbb{R}^4$ with an ALE manifold of A_{N-1} type. We will adopt on it the well-known Gibbons-Hawking multi-center metric [11].

Furthermore, we also study a flow solution involving Yang-Mills instantons turned on on the ALE space, thereby generalizing the solution discussed in [10]. Explicit instanton solutions on an ALE space can be written down for the $SU(2)$ gauge group [12], [13], [14] and we will then restrict ourselves to these solutions. The resulting supergravity solutions describe RG flows in two dimensional dual field theories and have asymptotic geometries $AdS_3 \times S^3/\mathbb{Z}_N$ in the UV and $AdS_3 \times S^3$ in the IR. The former arises from the limit where one goes to the boundary of the ALE, the latter when one zooms near one of the smooth ALE centers. Notice that in this case the solution describes the flow from a (2,0) UV CFT to a (4,0) IR CFT, contrary to the case of [10], where both fixed points were (4,0) CFT's. Indeed, in the UV we have $\mathbb{Z}_N$ projection, due to asymptotic topology of the ALE space.

We will then move to study flow solutions in 10D type IIB and type I$'$ theories (by the latter we mean IIB on $T^2/(-1)^{F_L}\Omega I_2$, the double T-dual of type I on T^2 [15]) on an ALE background. These solutions describe RG flows of four dimensional UV CFT's with $N = 2$ supersymmetry. In the type IIB case our solution is a variation on the theme discussed in [8, 9] for the ALE space of the form $\mathbb{C}^3/\mathbb{Z}_3$ and for the conifold, respectively, which describe flows from $N = 1$ to $N = 4$ CFT's. The RG flow in type IIB theory on $\mathbb{C}^3/\mathbb{Z}_3$ has also been studied in more details in [16], recently. Our flows interpolate between $N = 2$ quiver gauge theories with product gauge group in the UV and the $N = 4$ $SU(n)$ supersymmetric Yang-Mills theory in the IR. The corresponding asymptotic geometries are $AdS_5 \times S^5/\mathbb{Z}_N$ and $AdS_5 \times S^5$.

The discussion becomes more interesting in type I$'$ theory: in this case we find that the critical points are described by the geometries $AdS_5 \times S^5/(\mathbb{Z}_N \times \mathbb{Z}_2)$ in the UV and $AdS_5 \times S^5/\mathbb{Z}_2$ in the IR. The $\mathbb{Z}_2$ is identified with $(-1)^{F_L}\Omega I_2$. The UV gauge groups are more complicated than those of type IIB case, and are among the (unoriented)

quiver gauge groups discussed in [17]. The quiver diagrams have different structures depending on whether N is even or odd, and for N even there are in addition two possible projections, resulting in two different quiver structures. This is what will make the discussion of RG flows richer and more interesting. We will in fact verify the agreement between the geometric picture emerging from the supergravity solutions and the corresponding field theory description, where the flows are related to the Higgsing of the gauge group, i.e. they are driven by vacuum expectation values of scalar fields belonging to the hypermultiplets of the $N = 2$ theories.

We also consider more general RG flows, in which not all the UV gauge group is broken to a single diagonal IR subgroup. In other words, the IR theory can be another, smaller, quiver gauge theory. The associated flows are the flows between two $N = 2$ quiver gauge theories, and the corresponding geometries are given by $AdS_5 \times S^5/(\mathbb{Z}_N \times \mathbb{Z}_2)$ and $AdS_5 \times S^5/(\mathbb{Z}_M \times \mathbb{Z}_2)$ with $M < N$. We will find that field theory considerations do not allow all possible flows with arbitrary values of M and N and some symmetry breaking patterns are forbidden. Actually, we will see that these features are reproduced by the geometry, having to do with the fact that the Ω projection does not allow arbitrary ALE geometry since it projects out or identify the geometric moduli, as was already observed in a different context in [18]. In fact, we will obtain a very satisfactory agreement between with the field theory and the supergravity pictures.

The paper is organized as follows. In section 2, we present flow solutions in (1,0) six dimensional supergravity coupled to an anti-symmetric tensor multiplet on the ALE background. We then add $SU(2)$ instantons by coupling the supergravity theory to $SU(2)$ Yang-Mills multiplets and turning on the instantons on the ALE space. In section 3, we find supersymmetric solutions to type IIB and type I′ theories. Unfortunately, in this case, we are not able to obtain the explicit form of the solutions. However their existence, with the required boundary conditions are guaranteed on general mathematical grounds. The central charges along with field theory descriptions of the flows are also given. In section 4, we consider more general RG flows in which both the UV and IR CFTs are $N = 2$ quiver gauge theories and give a geometric interpretation for the symmetry breaking patterns. Finally, we make some conclusions and comments in section 5.

2. RG flows in six dimensional supergravity

In this section, we will find flows solution in (1,0) six dimensional supergravity. We begin with a review of (1,0) supergravity and focus mainly on relevant formulae we will use throughout this section. We proceed by studying an RG flow solution on the

ALE background and compute the ratio of the central charges of the UV and IR fixed points. We then include $SU(2)$ instantons on the ALE background. This is also a generalization of the solution studied in [10] in which the flow involves only Yang-Mills instantons. We will see that the result is a combined effect of gravitational instantons studied here and $SU(2)$ instantons studied in [10]. Finally, we discuss the left and right central charges with a subleading correction including curvature squared terms on the gravity side.

2.1 An RG flow with graviational instantons

We now study a supersymmetric RG flow solution in (1,0) six dimensional supergravity constructed in [19]. We are interested in the six dimensional supergravity theory coupled to one tensor multiplet and $SU(2)$ Yang-Mills multiplets. We refer the reader to [19] for the detailed construction of this theory. The equations of motion for bosonic fields are given by [19]

$$
\hat{R}_{MN} - \frac{1}{2}\hat{g}_{MN}\hat{R} - \frac{1}{3}e^{2\hat{\theta}}\left(3\hat{G}_{3MPQ}\hat{G}_{3N}{}^{PQ} - \frac{1}{2}\hat{g}_{MN}\hat{G}_{3PQR}\hat{G}_3^{PQR}\right)
$$

$$
-\partial_M\hat{\theta}\partial^M\hat{\theta} + \frac{1}{2}\hat{g}_{MN}\partial_P\hat{\theta}\partial^P\hat{\theta} - e^{\hat{\theta}}\left(2\hat{F}_M^{IP}\hat{F}_{NP}^I - \frac{1}{2}\hat{g}_{MN}\hat{F}_{PQ}^I\hat{F}^{IPQ}\right) = 0, \qquad (2.1)
$$

$$
\hat{D}(e^{2\hat{\theta}}\hat{*}\hat{G}_3) + \tilde{v}\hat{F}^I \wedge \hat{F}^I = 0, \qquad (2.2)
$$

$$
\hat{D}[(ve^{\hat{\theta}} + \tilde{v}e^{-\hat{\theta}})\hat{*}\hat{F}^I] - 2ve^{2\hat{\theta}}\hat{*}\hat{G}_3 \wedge \hat{F}^I + 2\tilde{v}\hat{*}\hat{G}_3 \wedge \hat{F}^I = 0, \qquad (2.3)
$$

$$
\hat{d}\hat{*}\hat{d}\hat{\theta} + (ve^{\hat{\theta}} + \tilde{v}e^{-\hat{\theta}})\hat{*}\hat{F}^I \wedge \hat{F}^I + 2e^{2\hat{\theta}}\hat{*}\hat{G}_3 \wedge \hat{G}_3 = 0 \qquad (2.4)
$$

with the Bianchi identity for $\hat{G}_3$ given by

$$
\hat{D}\hat{G}_3 = v\hat{F}^I \wedge \hat{F}^I. \qquad (2.5)
$$

$\hat{\theta}$ is the scalar field in the tensor multiplet. Indices $M, N, \ldots = 0, \ldots, 5$ label six dimensional coordinates, and $I, J, \ldots$ are adjoint indices of the corresponding Yang-Mills gauge group, $SU(2)$ in the present case. For both v and $\tilde{v}$ non-zero, there is no invariant Lagrangian, but the presence of the Lagrangian is not relevant for our discussion. As in [10], we also assume that both v and $\tilde{v}$ are positive, and the hatted fields are six-dimensional ones. We also need supersymmetry transformations of fermionic fields which, in this case, are the gravitino ψ_M, gauginos λ^I and the fermion in the tensor

multiplet χ. With fermions being zero, these transformations are given by [19]

$$\delta\psi_M = \hat{D}_M\epsilon + \frac{1}{24}e^{\hat{\theta}}\Gamma^{NPQ}\Gamma_M\hat{G}_{3NPQ}\epsilon, \tag{2.6}$$

$$\delta\lambda^I = \frac{1}{4}\Gamma^{MN}\hat{F}^I_{MN}\epsilon, \tag{2.7}$$

$$\delta\chi = \frac{1}{2}\Gamma^M\partial_M\hat{\theta}\epsilon - \frac{1}{12}e^{\hat{\theta}}\Gamma^{MNP}\hat{G}_{3MNP}\epsilon. \tag{2.8}$$

The metric ansatz is

$$ds^2_6 = e^{2f}(-dx^2_0 + dx^2_1) + e^{2g}ds^2_4. \tag{2.9}$$

The four dimensional metric ds^2_4 will be chosen to be the gravitational multi-instantons of [11]. This is an asymptotically locally Euclidean space (ALE) with the metric

$$ds^2_4 = V^{-1}(d\tau + \vec{\omega}.d\vec{x})^2 + Vd\vec{x}.d\vec{x}. \tag{2.10}$$

The function V is given by

$$V = \sum_{i=1}^{N}\frac{1}{|\vec{x} - \vec{x}_i|}. \tag{2.11}$$

The function $\vec{\omega}$ is related to V via

$$\vec{\nabla} \times \vec{\omega} = \vec{\nabla}V, \tag{2.12}$$

and the τ has period 4π. We also choose the gauge

$$\vec{\omega}.d\vec{x} = \sum_{i=1}^{N}\cos\theta_i d\phi_i \tag{2.13}$$

as in [20]. The point $\vec{x}_i$ is the origin of the spherical coordinates (r_i, θ_i, ϕ_i) with $r_i = |\vec{x} - \vec{x}_i|$. The procedure is now closely parallel to that of [10], so we only repeat the main results here and refer the reader to [10] for the full derivation. Although $\hat{A}^I = 0$ in the present case, it is more convenient to work with non-zero $\hat{A}^I$ since equations with non-zero $\hat{A}^I$ will be used later in the next subsection. We will keep $ds^2_4 = g_{\alpha\beta}dz^\alpha dz^\beta$, $\alpha = 2, 3, 4, 5$, in deriving all the necessary equations. The ansatz for $\hat{G}_3$ and $\hat{A}^I$, $I = 1, 2, 3$ are

$$\hat{A}^I = A^I, \qquad \hat{F}^I = F^I, \qquad \hat{G}_3 = G + dx_0 \wedge dx_1 \wedge d\Lambda. \tag{2.14}$$

As in [10], unhatted fields are four dimensional ones living on the four dimensional space with the metric ds^2_4 and depending only on z^α. Equation (2.2) gives

$$D(e^{2\theta-2f+2g} * d\Lambda) = vF^I \wedge F^I, \tag{2.15}$$

$$*G = e^{-2\theta-2f+2g}d\tilde{\Lambda} \tag{2.16}$$

where $\tilde{\Lambda}$ is a z^α dependent function. We now take F^I to be self dual with respect to four dimensional $*$. In order to solve the Killing spinor equations, we impose the chirality condition $\Gamma_{01}\epsilon = \epsilon$ which implies $\Gamma_{2345}\epsilon = \epsilon$ by the six dimensional chirality $\Gamma_7\epsilon = \epsilon$. So, equation $\delta\lambda^I = 0$ is trivially satisfied because $\Gamma_{\alpha\beta}$ is anti-selfdual. Furthermore, the condition $\Gamma_{01}\epsilon = \epsilon$ also breaks half of the (1,0) supersymmetry, so our flow solution preserves half of the eight supercharges. It is easy to show that $\delta\chi = 0$ and $\delta\psi_\mu = 0$ give, respectively,

$$\partial\!\!\!/\theta + e^{\theta-2f}\partial\!\!\!/\Lambda - e^{-\theta-2f}\partial\!\!\!/\tilde{\Lambda} = 0, \tag{2.17}$$

$$\Gamma_\mu\partial\!\!\!/f - \frac{1}{2}e^{\theta-2f}\Gamma_\mu\partial\!\!\!/\Lambda - \frac{1}{2}e^{-\theta-2f}\Gamma_\mu\partial\!\!\!/\tilde{\Lambda} = 0. \tag{2.18}$$

Taking combinations $(2.17) \pm (2.18)$, we find

$$\Lambda = \frac{1}{2}e^{-\theta+2f} + C_1, \qquad \tilde{\Lambda} = \frac{1}{2}e^{\theta+2f} + C_2 \tag{2.19}$$

with constants of integration C_1 and C_2. Equation $\delta\psi_\alpha = 0$ reads

$$D_\alpha\tilde{\epsilon} - \frac{1}{2}\Gamma_{\beta\alpha}\partial^\beta(f+g)\tilde{\epsilon} = 0. \tag{2.20}$$

We have used $\epsilon = e^{\frac{f}{2}}\tilde{\epsilon}$. Equation (2.20) can be satisfied provided that $g = -f$ and

$$D_\alpha\tilde{\epsilon} = 0. \tag{2.21}$$

The latter condition requires that $\tilde{\epsilon}$ is a Killing spinor on the ALE space. The ALE space has $SU(2)$ holonomy and admits two Killing spinors out of the four spinors. Therefore, the flow solution entirely preserves $\frac{1}{4}$ of the eight supercharges, or $N = 2$ in two dimensional langauge, along the flow.

In this subsection, we study only the effect of gravitational instantons, so we choose $A^I = 0$ from now on. Using (2.19), we can write (2.5) and (2.15) as

$$\Box e^{-\theta-2f} = 0 \qquad \text{and} \qquad \Box e^{\theta-2f} = 0. \tag{2.22}$$

The $\Box$ in these equations is the covariant scalar Laplacian on the ALE space

$$\Box = \frac{1}{V}[V^2\partial_\tau^2 + (\vec{\nabla} - \vec{\omega}\partial_\tau).(\vec{\nabla} - \vec{\omega}\partial_\tau)]. \tag{2.23}$$

Our flow is described by a simple ansatz as follows. We first choose $\theta = 0$. It is straightforward to check that all equations of motion as well as BPS equations are satisfied. We then have only a single equation to be solved

$$\Box e^{-2f} = 0. \tag{2.24}$$

We now choose f to be τ independent of the form

$$e^{-2f} = \frac{c}{|\vec{x} - \vec{x_1}|} \tag{2.25}$$

where c is a constant. This is clearly a solution of (2.24) since for τ independent functions, the $\Box$ reduce to the standard three dimensional Laplacian $\vec{\nabla}.\vec{\nabla}$. We will now show that this solution describes an RG flow between two fixed points given by $|\vec{x}| \to \infty$ and $\vec{x} \to \vec{x_1}$. We emphasize that the point $\vec{x_1}$ is purely conventional since any point x_i with $i = 1 \ldots N$ will work in the same way. Notice that for general τ dependent solution, the solution to the harmonic function will be given by the Green function on ALE spaces. The explicit form of this Green function will be given in the next subsection. Furthermore, with τ dependent solution, the IR fixed point of the flow can also be given by $\vec{x} \to \vec{y}$ where $\vec{y}$ is a regular point on the ALE space rather than one of the ALE canter $\vec{x_i}$. The crucial point in our discussion is the behavior of the Green function near the fixed points such that the geometry contains AdS_3. However, for the present case, we restrict ourselves to the ansatz (2.25).

When $|\vec{x}| \to \infty$, we have

$$e^{-2f} = \frac{c}{|\vec{x} - \vec{x_1}|} \to \frac{c}{\zeta}, \qquad \zeta \equiv |\vec{x}|,$$

$$V = \sum_{i=1}^{N} \frac{1}{|\vec{x} - \vec{x_i}|} \to \frac{N}{\zeta}. \tag{2.26}$$

In this limit, the ALE metric becomes

$$ds_4^2 = \frac{\zeta}{N}(d\tau + N \cos\theta d\phi)^2 + \frac{N}{\zeta}(d\zeta^2 + \zeta^2 d\Omega_2^2) \tag{2.27}$$

where we have written the flat three dimensional metric $d\vec{x}.d\vec{x}$ in spherical coordinates with the S^2 metric $d\Omega_2^2$. The factor $N \cos\theta d\phi$ arises from $\sum_{i=1}^{N} \cos\theta_i d\phi_i$ since in the limit $|\vec{x}| \to \infty$ all (θ_i, ϕ_i) are the same to leading order. By changing the coordinate ζ to r defined by $\zeta = \frac{r^2}{4N}$, we obtain

$$ds_4^2 = dr^2 + \frac{r^2}{4}\left[\left(\frac{d\tau}{N} + \cos\theta d\phi\right)^2 + d\Omega_2^2\right]. \tag{2.28}$$

The full six-dimensional metric is then given by

$$ds_6^2 = \frac{r^2}{4Nc}dx_{1,1}^2 + \frac{4Nc}{r^2}dr^2 + 4Nc\left[\left(\frac{d\tau}{N} + \cos\theta d\phi\right)^2 + d\Omega_2^2\right]. \tag{2.29}$$

The expression in the bracket is the metric on $S^3/\mathbb{Z}_N$. So, the six dimensional geometry is $AdS_3 \times S^3/\mathbb{Z}_N$ with the radii of AdS_3 and $S^3/\mathbb{Z}_N$ being $L_\infty = 2\sqrt{Nc}$.

When $\vec{x} \to \vec{x}_1$, we find

$$e^{-2f} = \frac{c}{\xi}, \qquad \xi \equiv |\vec{x} - \vec{x}_1|,$$

$$V = \sum_{i=1}^{N} \frac{1}{|\vec{x} - \vec{x}_i|} = \frac{1}{\xi}. \tag{2.30}$$

The ALE metric becomes

$$ds_4^2 = \xi(d\tau + \cos\theta_1 d\phi_1)^2 + \frac{1}{\xi}(d\xi^2 + \xi^2 d\Omega_2^2). \tag{2.31}$$

In this limit, $\sum_{i=1}^{N} \cos\theta_i d\phi_i \sim \cos\theta_1 d\phi_1$ to leading order. Writing $\xi = \frac{r^2}{4}$, we obtain

$$ds_4^2 = dr^2 + \frac{r^2}{4}[(d\tau - \cos\theta_1 d\phi_1)^2 + d\Omega_2^2] \tag{2.32}$$

which is the metric on $\mathbb{R}^4$. The six-dimensional metric now takes the form

$$ds_6^2 = \frac{r^2}{4c}dx_{1,1}^2 + \frac{4c}{r^2}dr^2 + 4c d\Omega_3^2 \tag{2.33}$$

where $d\Omega_3^2$ is the metric on S^3. This geometry is $AdS_3 \times S^3$ with AdS_3 and S^3 having the same radius $2\sqrt{c}$. The central charge of the dual CFT is given by

$$c = \frac{3L}{2G_N^{(3)}}. \tag{2.34}$$

We find the ratio of the central charges

$$\frac{c_1}{c_\infty} = \frac{L_1 G_{N\infty}^{(3)}}{L_\infty G_{N1}^{(3)}} = \frac{L_1 \text{Vol}(S^3)}{L_\infty \text{Vol}(S^3/\mathbb{Z}_N)}$$

$$= N \left(\frac{L_1}{L_\infty}\right)^4 = \frac{1}{N} \tag{2.35}$$

where we have used $G_N^{(3)} = \frac{G_N^{(6)}}{\text{Vol}(M)}$ for six-dimensional theory compactified on a compact space M. The flow interpolates between $AdS_3 \times S^3/\mathbb{Z}_N$ in the UV to $AdS_3 \times S^3$ in the IR. The UV CFT has (2,0) supersymmetry because of the $\mathbb{Z}_N$ projection, so our flow describes an RG flow from the (2,0) CFT in the UV to the (4,0) CFT in the IR.

We now consider the central charge on the gravity side including the curvature squared terms. The bulk gravity is three dimensional, and the Riemann tensor can

be written in terms of the Ricci tensor and Ricci scalar. To study the effect of higher derivative terms, we add the $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ term to the (1,0) six dimensional action. The supersymmetrization of this term has been studied in [21]. We temporarily drop the hat to simplify the expressions. The Lagrangian with the auxiliary fields integrated out is given by [22]

$$\mathcal{L} = \sqrt{-g}e^{-2\theta}\left[R + 4\partial_\mu\theta\partial^\mu\theta - \frac{1}{12}G_3^{\mu\nu\rho}G_{3\mu\nu\rho}\right] + \frac{1}{4}\alpha\sqrt{-g}\tilde{R}^{\mu\nu\rho\sigma}\tilde{R}_{\mu\nu\rho\sigma}$$

$$+\frac{1}{16}\beta\epsilon^{\mu\nu\rho\sigma\tau\lambda}\tilde{R}^{\alpha\beta}{}_{\mu\nu}\tilde{R}_{\alpha\beta\rho\sigma}b_{\tau\lambda} \tag{2.36}$$

where $\tilde{R}_{\mu\nu\rho\sigma}$ is computed with the modified connection $\tilde{\Gamma}^\rho{}_{\mu\nu} = \Gamma^\rho{}_{\mu\nu} - \frac{1}{2}G^\rho_{3\mu\nu}$. The $b_{\lambda\tau}$ is the two-form field whose field strength is G_3. Reducing (2.36) on S^3 with $G_3 = 2S\epsilon_3 + 2m\omega_3$ where ϵ_3 and ω_3 are volume forms on ds_3^2 and S^3, respectively gives [22]

$$e^{-1}\mathcal{L} = e^{-2\theta}(R + 4\partial_\mu\theta\partial^\mu\theta + 4m^2 + 2S^2) + 4mS$$

$$-2\beta m\left[RS + 2S^3 - \frac{1}{4}\epsilon^{\mu\nu\rho}\left(R^{ab}{}_{\mu\nu}\omega_{\rho ab} + \frac{2}{3}\omega_\mu{}^a{}_b\omega_\nu{}^b{}_c\omega_\rho{}^c{}_a\right)\right]$$

$$+\frac{1}{4}\alpha(4R^{\mu\nu}R_{\mu\nu} - R^2 - 8\partial_\mu S\partial^\mu S + 12S^4 + 4RS^2). \tag{2.37}$$

As shown in [22], $S = -m$ on the AdS_3 background, and m is related to the AdS radius via $m = \frac{1}{L}$. The left and right moving central charges can be computed as in [23, 24]. The result is [22]

$$c_L = \frac{3L}{2G_N^{(3)}}\left(1 + \frac{4\beta}{L^2}\right), \qquad c_R = \frac{3L}{2G_N^{(3)}}. \tag{2.38}$$

We find that

$$\text{UV}: \qquad c_L = \frac{48\pi^2c^2N}{G_N^{(6)}}\left(1 + \frac{\beta}{cN}\right), \qquad c_R = \frac{48\pi^2c^2N}{G_N^{(6)}}, \tag{2.39}$$

$$\text{IR}: \qquad c_L = \frac{48\pi^2c^2}{G_N^{(6)}}\left(1 + \frac{\beta}{c}\right), \qquad c_R = \frac{48\pi^2c^2}{G_N^{(6)}}. \tag{2.40}$$

We end this subsection by finding the dimension of the dual operator driving the flow. This is achieved by expanding the metric around the UV fixed point, $|\vec{x}| \to \infty$ in our solution. e^{-2f} and V can be expanded as

$$e^{-2f} = \frac{c}{|\vec{x} - \vec{x}_1|} \sim \frac{1}{\zeta}\left(1 + \frac{a_1\cos\varphi_1}{\zeta} - \frac{a_1^2}{2\zeta^2}\right) + \dots,$$

$$V = \sum_{i=1}^{N}\frac{1}{|\vec{x} - \vec{x}_i|} \sim \frac{N}{\zeta} + \sum_{i=1}^{N}\left(\frac{a_i\cos\varphi_i}{\zeta^2} - \frac{a_i^2}{2\zeta^3}\right) + \dots \tag{2.41}$$

where φ_i are angles between $\vec{x}$ and $\vec{x}_i$. We have also defined $\zeta \equiv |\vec{x}|$ and $a_i \equiv |\vec{x}_i|$. By substituting (2.41) in (2.9), it is then straightforward to obtain the behavior of the metric fluctuation which is of order $\mathcal{O}(r^{-2})$. This gives $\Delta = 2$ indicating that the flow is driven by a vacuum expectation value of a marginal operator.

2.2 An RG flow with gravitational and $SU(2)$ Yang-Mills instantons

We now add Yang-Mills instantons to the solution given in the previous subsection. This involves constructing instantons on ALE spaces. Some explicit instantons solutions on an ALE space are given in [12]. We are interested in $SU(2)$ instantons whose explicit solutions can be written down. The solution can be expressed in the form [12]

$$A_\alpha^I dx^\alpha = -\eta_{ab}^I e^a E^b \ln H . \tag{2.42}$$

The vielbein e_α^a and its inverse E_a^α for the metric (2.10) are given by

$$e^0 = V^{-\frac{1}{2}}(d\tau + \vec{\omega}.d\vec{x}), \qquad e^l = V^{\frac{1}{2}}dx^l, \tag{2.43}$$

$$E_0 = V^{\frac{1}{2}}\frac{\partial}{\partial\tau}, \qquad E_l = V^{-\frac{1}{2}}\left(\frac{\partial}{\partial x^l} - \omega_l\frac{\partial}{\partial\tau}\right). \tag{2.44}$$

The η_{ab}^I's are the usual 't Hooft tensors and $l = 1, 2, 3$. This form resembles the $SU(2)$ instantons on the flat space $\mathbb{R}^4$. Self duality of F^I requires that H satisfies the harmonic equation on the ALE space

$$\nabla_a\nabla^a H = 0 . \tag{2.45}$$

The solution is given by

$$H = H_0 + \sum_{j=1}^n \lambda_j G(x, y_j) \tag{2.46}$$

where H_0 and λ_j are constants, and $G(x, y_j)$ is the Green's function on the ALE space given in [20] with $x = (\tau, \vec{x})$. Its explicit form is

$$G(x, x') = \frac{\sinh U}{16\pi^2|\vec{x} - \vec{x}'|(\cosh U - \cos T)} \tag{2.47}$$

where

$$U(x, x') = \frac{1}{2}\sum_{i=1}^N \ln\left(\frac{r_i + r_i' + |\vec{x} - \vec{x}'|}{r_i + r_i' - |\vec{x} - \vec{x}'|}\right), \qquad r_i = |\vec{x} - \vec{x}_i|,$$

$$T(x, x') = \frac{1}{2}(\tau - \tau') + \sum_{i=1}^N \tan^{-1}\left[\tan\left[\frac{\phi_i - \phi_i'}{2}\right]\frac{\cos\frac{\theta_i+\theta_i'}{2}}{\cos\frac{\theta_i-\theta_i'}{2}}\right]. \tag{2.48}$$

This solution is obviously τ dependent and can be thought of as a generalization of the τ independent solution of [13]. The latter is subject to the constraint $n \leq N$ since the finite action requires that the instantons must be put at the ALE centers. We emphasize here that the $\vec{y}_j$ inside the y_j in (2.46) needs not necessarily coincide with the ALE center $\vec{x}_i$. Therefore, $\vec{y}_j$ could be any point, ALE center or regular point, on the ALE space. However, in our flow solution given below, we will choose one of the $\vec{y}_j$'s to coincide with one of the ALE centers $\vec{x}_i$'s which is, by our convention, chosen to be $\vec{x}_1$.

As in the flat space case, we can write

$$F_{ab}^I F^{Iab} = -4\square\square \ln H \tag{2.49}$$

which can be shown by using the properties of η_{ab}^I given in [25] and the fact that H is a harmonic function on the ALE space as well as the Ricci flatness of the ALE space. Using this relation, we obtain

$$*(F^I \wedge F^I) = *(*F^I \wedge F^I) = \frac{1}{2}F_{ij}^I F^{Iij} = -2\square\square \ln H . \tag{2.50}$$

Equations (2.5) and (2.15) become

$$\square\, e^{-\theta-2f} = 4v\square\square \ln H, \tag{2.51}$$

$$\square\, e^{\theta-2f} = 4\tilde{v}\square\square \ln H . \tag{2.52}$$

The solutions to these equations are of the form

$$e^{-\theta-2f} = f_1 + 4v\square \ln H,$$
$$e^{\theta-2f} = f_2 + 4\tilde{v}\square \ln H \tag{2.53}$$

where f_1 and f_2 are solutions to the homogeneous equations. The Green function $G(x, x')$ in (2.47) is singular when $x \sim x'$. The behavior of $G(x, x')$ in this limit is [20]

$$G(x, x') = \frac{1}{4\pi^2 |x - x'|^2} \tag{2.54}$$

where

$$|x - x'|^2 = V|\vec{x} - \vec{x}'|^2 + V^{-1}[\tau - \tau' + \vec{\omega}.(\vec{x} - \vec{x}')]^2. \tag{2.55}$$

We remove this singularity, in our case $x' \sim y_j$, from our solution by adding $G(x, y_j)$, with appropriate coefficients, to (2.53). We also choose f_1 and f_2 to be $\frac{c}{|\vec{x}-\vec{x}_1|}$ and $\frac{d}{|\vec{x}-\vec{x}_1|}$,

respectively. This choice is analogous to the solution in the previous subsection with c and d being constants. Collecting all these, we find

$$e^{-\theta-2f} = \frac{c}{|\vec{x} - \vec{x}_1|}$$
$$+ 4v \left[\Box \ln \left(H_0 + \sum_{j=1}^{n} \lambda_j G(x, y_j) \right) + 16\pi^2 \sum_{j=1}^{n} G(x, y_j) \right], \qquad (2.56)$$

$$e^{\theta-2f} = \frac{d}{|\vec{x} - \vec{x}_1|}$$
$$+ 4\tilde{v} \left[\Box \ln \left(H_0 + \sum_{j=1}^{n} \lambda_j G(x, y_j) \right) + 16\pi^2 \sum_{j=1}^{n} G(x, y_j) \right]. \qquad (2.57)$$

The metric warp factor e^{-2f} can be obtained by multiplying (2.56) and (2.57). We now study the behavior of this function in the limits $\vec{x} \to \vec{x}_1$ and $|\vec{x}| \to \infty$.

As $\vec{x} \to \vec{x}_1$, the terms involving $G(x, x_1)$ in the square bracket in (2.56) and (2.57) do not contribute since the poles of the two terms cancel each other. The other terms involving $G(x, y_j)$, $\vec{y}_j \neq \vec{x}_1$, are subleading compared to f_1 and f_2. We find

$$e^{-\theta-2f} = \frac{d}{|\vec{x} - \vec{x}_1|}, \qquad e^{\theta-2f} = \frac{c}{|\vec{x} - \vec{x}_1|} \qquad (2.58)$$

or

$$e^{-2f} = \frac{\sqrt{cd}}{|\vec{x} - \vec{x}_1|}. \qquad (2.59)$$

By using the coordinate changing as in the previous subsection $|\vec{x} - \vec{x}_1| = \frac{r^2}{4}$, it can be shown that the metric is of the form of $AdS_3 \times S^3$

$$ds_6^2 = \frac{r^2}{4\sqrt{cd}} dx_{1,1}^2 + \frac{4\sqrt{cd}}{r^2} dr^2 + 4\sqrt{cd} d\Omega_3^2. \qquad (2.60)$$

As $|\vec{x}| \to \infty$, the Green function (2.47) becomes

$$G(x, x') = \frac{1}{16\pi^2 |\vec{x} - \vec{x}'|} \qquad (2.61)$$

because U defined in (2.48) becomes infinite. We find

$$e^{-\theta-2f} = \frac{c}{|\vec{x} - \vec{x}_1|} + 4v \sum_{i=1}^{n} \frac{1}{|\vec{x} - \vec{y}_i|} \sim \frac{c + 4vn}{|\vec{x}|},$$

$$e^{\theta-2f} = \frac{d}{|\vec{x} - \vec{x}_1|} + 4\tilde{v} \sum_{i=1}^{n} \frac{1}{|\vec{x} - \vec{y}_i|} \sim \frac{d + 4\tilde{v}n}{|\vec{x}|}. \qquad (2.62)$$

The warp factor is now given by

$$e^{-2f} = \frac{\sqrt{(c+4nv)(d+4n\tilde{v})}}{|\vec{x}|} \,. \tag{2.63}$$

The six-dimensional metric becomes $AdS_3 \times S^3/\mathbb{Z}_N$, with $|\vec{x}| = \frac{r^2}{4N}$,

$$ds_6^2 = \frac{r^2}{\ell^2}dx_{1,1}^2 + \frac{\ell^2}{r^2}dr^2 + \ell^2 \left[\left(\frac{d\tau}{N} + \cos\theta d\phi\right)^2 + d\Omega_2^2 \right] \tag{2.64}$$

where the AdS_3 radius is given by

$$\ell = 2\sqrt{N}[(c+4nv)(d+4n\tilde{v})]^{\frac{1}{4}} \,. \tag{2.65}$$

The ratio of the central charges can be found in the same way as that in the previous subsection and is given by

$$\frac{c_1}{c_\infty} = N\left(\frac{L_1}{L_\infty}\right)^4 = \frac{cd}{N(c+4nv)(d+4n\tilde{v})} \,. \tag{2.66}$$

For $N=1$, the ALE space becomes a flat $\mathbb{R}^4$, and we obtain the result given in [10]. As in the previous subsection, the solution describes an RG flow from a (2,0) CFT to a (4,0) CFT in the IR. The central charges to curvature squared terms are given by

$$\text{UV}: c_L = \frac{48\pi^2(c+4nv)(d+4n\tilde{v})N}{G_N^{(6)}}\left(1 + \frac{\beta}{N\sqrt{(c+4nv)(d+4n\tilde{v})}}\right),$$

$$c_R = \frac{48\pi^2(c+4nv)(d+4n\tilde{v})N}{G_N^{(6)}}, \tag{2.67}$$

$$\text{IR}: c_L = \frac{48\pi^2 cd}{G_N^{(6)}}\left(1 + \frac{\beta}{cd}\right), \qquad c_R = \frac{48\pi^2 cd}{G_N^{(6)}} \,. \tag{2.68}$$

As in the previous subsection, it can be shown that this is also a vev flow driven by a vev of a marginal operator of dimension two.

3. RG flows in type IIB and type I′ theories

In this section, we study an RG flow solution in type IIB theory on an ALE background. Since there is no gauge field in type IIB theory, the corresponding flow solution only involves gravitational instantons. We also consider a solution in type I′ theory which is a T-dual of the usual type I theory on T^2 and can also be obtained from type IIB theory on $T^2/(-1)^{F_L}\Omega I_2$. As we will see, in type I′ theory, there are more possibilities of the gauge groups for the quiver gauge theory in the UV and, as a result, more possible RG flows.

3.1 RG flows in type IIB theory

We now study a supersymmetric flow solution in type IIB theory. We begin with supersymmetry transformations of the gravitino ψ_M and the dilatino χ. These can be found in various places, see for example [26, 27], and are given by

$$\delta\chi = iP_M\Gamma^M\epsilon^* - \frac{i}{24}F_{M_1M_2M_3}\Gamma^{M_1M_2M_3}\epsilon,$$

$$\delta\psi_M = \nabla_M\epsilon - \frac{i}{1920}F^{(5)}_{M_1M_2M_3M_4M_5}\Gamma^{M_1M_2M_3M_4M_5}\Gamma_M\epsilon$$

$$+\frac{1}{96}F_{M_1M_2M_3}(\Gamma_M{}^{M_1M_2M_3} - 9\delta_M^{M_1}\Gamma^{M_2M_3})\epsilon \tag{3.1}$$

where

$$P_M = \frac{1}{2}(\partial_M\phi + ie^\phi\partial_M C_0),$$

$$F_{M_1M_2M_3} = e^{-\frac{\phi}{2}}H_{M_1M_2M_3} + ie^{\frac{\phi}{2}}F_{M_1M_2M_3}. \tag{3.2}$$

In our ansatz, we choose $\phi = 0$, $C_0 = 0$ and $F_{M_1M_2M_3} = 0$, so $\delta\chi = 0$ is automatically satisfied. The ten dimensional metric is given by

$$ds^2 = e^{2f}dx_{1,3}^2 + e^{2g}ds_4^2 + e^{2h}(dr^2 + r^2d\theta^2). \tag{3.3}$$

The metric ds_4^2 is the ALE metric in (2.10), and the functions f, g and h depend only on ALE coordinates y^a and r. We will use indices $\mu, \nu = 0, \ldots, 3$, $a, b = 4, \ldots, 8$. The ansatz for the self-dual five-form field strength is

$$F^{(5)} = \tilde{F} + \hat{*}\tilde{F} \tag{3.4}$$

where $\hat{*}$ is the ten dimensional Hodge duality. We choose $\tilde{F}$ to be

$$\tilde{F} = dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \wedge (U^{(1)} + Kdr) + rG^{(3)} \wedge dr \wedge d\theta + r\tilde{G}^{(4)} \wedge d\theta,$$

$$\hat{*}\tilde{F} = e^{-4f}(e^{2(g+h)} * U^{(1)} + e^{4g} * Krd\theta) + e^{4f-2(g+h)}dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \wedge *G^{(3)}$$

$$+e^{4(f-g)} * \tilde{G}^{(4)}dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \wedge dr \tag{3.5}$$

with all functions depending only on y^a and r. We have used the convention $\epsilon_{01234567r\theta} = 1$. The notation $X^{(n)}$ means the n-form X on the four dimensional space whose metric is ds_4^2. The Bianchi identity $DF^{(5)} = 0$ and self duality condition impose the conditions

$$dU^{(1)} = 0, \qquad dK = \partial_r U^{(1)} \Rightarrow U^{(1)} = d\Lambda, \qquad K = \partial_r\Lambda + c_1,$$

$$*G^{(3)} = e^{-4f+2(g+h)}d\Lambda, \qquad *\tilde{G}^{(4)} = e^{-4(f-g)}K. \tag{3.6}$$

The $*$ and d are the Hodge dual and exterior derivative on ds_4^2, and c_1 is a constant.

From (3.3), we can read off the vielbein components

$$e^{\hat{\mu}} = e^f dx^\mu, \qquad e^{\hat{a}} = e^g \bar{e}^{\hat{a}}, \qquad e^{\hat{r}} = e^h dr, \qquad e^{\hat{\theta}} = e^h r d\theta. \tag{3.7}$$

The $\bar{e}^{\hat{a}}$ is the vielbein on the ALE space. The spin connections are given by

$$\omega^{\hat{\theta}}_{\ \hat{r}} = e^{-h}\left(\frac{1}{r} + h'\right)e^{\hat{\theta}}, \qquad \omega^{\hat{\theta}}_{\ \hat{a}} = e^{-g}\partial_a h e^{\hat{\theta}}, \qquad \omega^{\hat{r}}_{\ \hat{a}} = e^{-g}\partial_a h e^{\hat{r}} - e^{-h}g'e^{\hat{a}},$$

$$\omega^{\hat{a}}_{\ \hat{b}} = e^{-g}(\partial_b g \delta^a_c - \partial^a g \delta^b_c)e^{\hat{c}} + e^{-g}\bar{\omega}^{\hat{a}}_{\ \hat{b}}, \qquad \omega^{\hat{\mu}}_{\ \hat{a}} = e^{-g}\partial_a f e^{\hat{\mu}},$$

$$\omega^{\hat{\mu}}_{\ \hat{r}} = e^{-h}f'e^{\hat{\mu}} \tag{3.8}$$

where $\bar{\omega}^{\hat{a}}_{\ \hat{b}}$ are spin connections on the ALE space. We also use the following ten dimensional gamma matrices

$$\Gamma_{\hat{\mu}} = \gamma_{\hat{\mu}} \otimes \mathbf{I}_4 \otimes \mathbf{I}_2, \qquad \Gamma_{\hat{a}} = \tilde{\gamma}_5 \otimes \gamma_{\hat{a}} \otimes \mathbf{I}_2,$$

$$\Gamma_{\hat{r}} = \tilde{\gamma}_5 \otimes \hat{\gamma}_5 \otimes \sigma_1, \qquad \Gamma_{\hat{\theta}} = \tilde{\gamma}_5 \otimes \hat{\gamma}_5 \otimes \sigma_2. \tag{3.9}$$

Throughout this paper, we use the notation $\mathbf{I}_n$ for $n \times n$ identity matrix. The chirality condition on ϵ is $\Gamma_{11}\epsilon = \tilde{\gamma}_5 \otimes \hat{\gamma}_5 \otimes \sigma_3 \epsilon = \epsilon$. $\tilde{\gamma}_5 = i\gamma_0\gamma_1\gamma_2\gamma_3$ and $\hat{\gamma}_5 = \gamma_4\gamma_5\gamma_6\gamma_7$ are chirality matrices in x^μ and y^a spaces, respectively. With only 5-form turned on, the relevant BPS equations come from

$$\delta\psi_M = \nabla_M \epsilon - \frac{i}{1920}\slashed{F}^{(5)}\Gamma_M \epsilon \tag{3.10}$$

where $\slashed{F}^{(5)} = F^{(5)}_{M_1 M_2 M_3 M_4 M_5}\Gamma^{M_1 M_2 M_3 M_4 M_5}$. It is now straightforward to show that all the BPS equations are satisfied provided that we choose

$$h = g = -f, \qquad \Lambda = 2e^{4f}, \qquad \epsilon = e^{\frac{1}{2}f + \frac{i}{2}\sigma_3\theta}\hat{\epsilon} \tag{3.11}$$

with $\hat{\epsilon}$ being the Killing spinor on the ALE space and satisfying the condition

$$\bar{\nabla}_a \hat{\epsilon} = 0. \tag{3.12}$$

Furthermore, $\hat{\epsilon}$ satisfies a projection condition $\gamma_5\hat{\epsilon} = \hat{\epsilon}$. So, the solution is again $\frac{1}{4}$ supersymmetric along the flow. With these conditions inserted in (3.4), we obtain the 5-form field

$$F^{(5)} = 2dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \wedge \hat{d}\Lambda + 2e^{-8f}\hat{*}\hat{d}\Lambda \tag{3.13}$$

where now $\hat{*}$ and $\hat{d}$ are those on the six dimensional space ALE $\times \mathbb{R}^2$ with coordinates (y^a, r, θ). Equation $DF^{(5)} = 0$ then gives

$$\hat{d}(e^{-8f}\hat{*}\hat{d}\Lambda) = \hat{d}\hat{*}\hat{d}e^{-4f} = 0. \tag{3.14}$$

So, the function e^{-4f} satisfies a harmonic equation on ALE$\times\mathbb{R}^2$.

It turns out to be difficult to find the explicit form of this harmonic function. This function can be constructed from the Green's function whose existence has been shown in [28], see also [29]. We now consider the behavior of this funciton at the two fixed points. The six dimensional metric is given by

$$d\tilde{s}^2 = V^{-1}(d\tau + \vec{\omega}.d\vec{x})^2 + V d\vec{x}.d\vec{x} + dr^2 + r^2 d\theta^2 \ . \tag{3.15}$$

As $|\vec{x}| \to \infty$, with the coordinate changing given in the previous section, the ALE metric become $\mathbb{R}^4/\mathbb{Z}_N$. So, the metric of the whole six dimensional space can be written as

$$d\tilde{s}^2 = dR^2 + R^2 ds^2(S^5/\mathbb{Z}_N) \tag{3.16}$$

where $R^2 = 4N|\vec{x}| + r^2$.

Similarly, we can show that as $\vec{x} \to \vec{x}_1$, the metric becomes the flat $\mathbb{R}^6$ metric

$$d\tilde{s}^2 = d\tilde{R}^2 + \tilde{R}^2 d\Omega_5^2 \tag{3.17}$$

where $\tilde{R}^2 = 4|\vec{x} - \vec{x}_1| + r^2$.

So, in order to interpolate between two conformal fixed points, this function must satisfy the boundary condition

$$e^{-4f} \sim \frac{1}{R^4} \ . \tag{3.18}$$

at both ends. There is also a relative factor of N between the two end points. This is due to the fact that the integral of the harmonic equation (3.14) on $d\tilde{s}^2$ must vanish, and this integral is in turn reduced to the integral of the gradient of the Green's function over S^5 and $S^5/\mathbb{Z}_N$ at the two end points. So, with all these requirements, the required harmonic function has boundary conditions

$$\begin{aligned}\vec{x} \to \vec{x}_1 : &\qquad e^{-4f} = \frac{C}{R^4}, \\ |\vec{x}| \to \infty : &\qquad e^{-4f} = \frac{CN}{R^4} \ .\end{aligned} \tag{3.19}$$

The full metrics at both end points take the form

$$\begin{aligned}\vec{x} \to \vec{x}_1 : &\qquad ds_{10}^2 = \frac{R^2}{\sqrt{C}}dx_{1,3}^2 + \frac{\sqrt{C}}{R^2}dR^2 + \sqrt{C}d\Omega_5^2 \\ |\vec{x}| \to \infty : &\qquad ds_{10}^2 = \frac{\tilde{R}^2}{\sqrt{NC}}dx_{1,3}^2 + \frac{\sqrt{NC}}{\tilde{R}^2}d\tilde{R}^2 + \sqrt{NC}ds^2(S^5/\mathbb{Z}_N).\end{aligned} \tag{3.20}$$

We obtain the two AdS_5 radii $L_1 = C^{\frac{1}{4}}$ and $L_\infty = (CN)^{\frac{1}{4}}$. The central charge is given by [30]

$$a = c = \frac{\pi L^3}{8 G_N^{(5)}}.$$ (3.21)

The ratio of the central charges is given by

$$\frac{a_1}{a_\infty} = \frac{c_1}{c_\infty} = \frac{L_1^8 \text{Vol}(S^5)}{L_\infty^8 \text{Vol}(S^5/\mathbb{Z}_N)} = N \left(\frac{L_1}{L_\infty}\right)^8 = \frac{1}{N}.$$ (3.22)

The flow describes the deformation of $N = 2$ quiver $SU(n)^N$ gauge theory in the UV to $N = 4$ $SU(n)$ SYM in the IR in which the gauge group $SU(n)$ is the diagonal subgroup of $SU(n)^N$.

We now compute the central charges to curvature squared terms. Higher derivative corrections to the central charges in four dimensional CFTs have been considered in many references, see for example [30, 31, 32]. The five dimensional gravity Lagrangian with higher derivative terms can be written as

$$\mathcal{L} = \frac{\sqrt{-g}}{2\kappa_5^2}(R + \Lambda + \alpha R^2 + \beta R_{\mu\nu} R^{\mu\nu} + \gamma R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma}).$$ (3.23)

Λ is the cosmological constant. The central charges a and c appear in the trace anomaly

$$\langle T^\mu_{\ \mu} \rangle = \frac{c}{16\pi^2} \left(R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 2 R_{\mu\nu} R^{\mu\nu} + \frac{1}{3} R^2 \right)$$
$$- \frac{a}{16\pi^2} (R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 4 R_{\mu\nu} R^{\mu\nu} + R^2).$$ (3.24)

Compare this with the holographic Weyl anomaly gives

$$a = \frac{\pi L^3}{8 G_N^{(5)}} \left[1 - \frac{4}{L^4}(10\hat{\alpha} + 2\hat{\beta} + \hat{\gamma}) \right],$$
$$c = \frac{\pi L^3}{8 G_N^{(5)}} \left[1 - \frac{4}{L^4}(10\hat{\alpha} + 2\hat{\beta} - \hat{\gamma}) \right]$$ (3.25)

where we have separated the AdS_5 radius out of $\alpha = \frac{\hat{\alpha}}{L^2}$, $\beta = \frac{\hat{\beta}}{L^2}$ and $\gamma = \frac{\hat{\gamma}}{L^2}$. Only γ can be determined from string theory calculation. Furthermore, there is an ambiguity in α and β due to field redefinitions.

For $N = 4$ SYM with gauge group $SU(n)$ in the IR, there is no correction from $R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma}$ term. To this order, the central charges are then given by

$$a_{IR} = c_{IR} = \frac{\pi^4 L^8}{8 G_N^{(10)}} = \frac{\pi^4 C^2}{8 G_N^{(10)}}.$$ (3.26)

On the other hand, in the UV, we have $N = 2$, $SU(n)^N$ quiver gauge theory. The central charges are

$$a_{UV} = \frac{\pi^4 N C^2}{8 G_N^{(10)}} \left[1 - \frac{4}{NC}(10\hat{\alpha} + 2\hat{\beta} + \hat{\gamma}) \right],$$

$$c_{UV} = \frac{\pi^4 N C^2}{8 G_N^{(10)}} \left[1 - \frac{4}{NC}(10\hat{\alpha} + 2\hat{\beta} - \hat{\gamma}) \right]. \tag{3.27}$$

The constant C in our solution is related to the number of D3-branes, N_3. The leading term in a and c is of order C^2 while the subleading one is of order C as expected. The analysis of the metric fluctuation can be carried out as in the six dimensional case and gives $\Delta = 2$. The flow is a vev flow driven by a vacuum expectation value of a relevant operator of dimension two.

Before discussing the RG flow on the dual field theory, let us recall that ALE gravitational instantons admit a hyperkahler quotient construction, which can be understood nicely in terms of the moduli space of a transverse (regular) D-brane probe moving off the orbifold fixed point in $\mathbb{R}^4/\mathbb{Z}_N$ [17], [18]. Starting with $U(N)$ valued fields X, $\bar{X}$ on which one performs the $\mathbb{Z}_N$ projection, one denotes the invariant (one-dimensional) components by $X_{i,i+1}$, $\bar{X}_{i+1,i}$, for $i = 0, \ldots, N-2$, $X_{N-1,0}$, $\bar{X}_{0,N-1}$, which are the links of the quiver diagram corresponding to the A_{N-1} extended Dynkin diagram. The resulting gauge group is $U(1)^N$, with a trivially acting center of mass $U(1)$. It is convenient to introduce the doublet fields Φ_r

$$\Phi_r = \begin{pmatrix} X_{r-1,r} \\ \bar{X}_{r,r-1}^\dagger \end{pmatrix} \tag{3.28}$$

for $r = 1, \ldots, N-1$, and

$$\Phi_0 = \begin{pmatrix} X_{N-1,0} \\ \bar{X}_{0,N-1}^\dagger \end{pmatrix} \tag{3.29}$$

After removing the trivial center of mass $U(1)$, the gauge group is $U(1)^{N-1}$, and the Φ's have definite charges with respect to it. After introducing Fayet-Iliopoulos (FI) terms $\vec{D}_r$, $r = 0, \ldots, N-1$, with $\sum_r \vec{D}_r = 0$, corresponding to closed string, blowing-up mouduli, one gets the following potential:

$$U = \sum_{r=0}^{N-1} \left(\Phi_r^\dagger \vec{\sigma} \Phi_r - \Phi_{r+1}^\dagger \vec{\sigma} \Phi_{r+1} + \vec{D}_r \right)^2. \tag{3.30}$$

and the $N-1$ independent D-flatness conditions, are then given by:

$$\Phi_{r+1}^\dagger \vec{\sigma} \Phi_{r+1} - \Phi_r^\dagger \vec{\sigma} \Phi_r = \vec{D}_r. \tag{3.31}$$

The ALE metric (2.10) can be obtained after defining the ALE coordinate and centers

$$\vec{x} = \Phi_0^\dagger \vec{\sigma} \Phi_0, \qquad \vec{x}_i = \sum_{r=0}^{i-1} \vec{D}_r, \tag{3.32}$$

respectively, and computing the gauge invariant kinetic term on the Φ's, subject to the D-terms constraints [18].

This procedure can be generalized to the case of n regular D3-branes transverse to the ALE space. Starting with $U(nN)$ valued Chan-Paton factors, the resulting theory after projection, is the $N = 2$ $SU(n)^N$ gauge theory, with hypermultiplets formed by the fields X_{ij} and $\bar{X}_{ij}$ related to the links of the quiver diagram as above, but now in the $(n, \bar{n})$, $(\bar{n}, n)$ representations of the $SU(n)$'s at the vertices of the quiver diagram. In addition, there are adjoint scalars W_i in the adjoint of $SU(n)$, belonging to the vector multiplets. The theory is conformally invariant and describes the dual $N = 2$ SCFT at the UV point.

In order to match with the RG flow from the UV to IR described previously on the gravity side, which gives an $N = 4$ theory in the IR, we consider the Higgs branch of the $N = 2$ UV theory discussed above. Therefore, we set $\langle W_i \rangle = 0$ and give vev's to the hypermultiplets X_{ij}, $\bar{X}_{ij}$. The equations governing the vacua of the theory are then the obvious matrix generalization of (3.31), with $N - 1$ independent triplets of FI terms for the $N - 1$ $U(1)$'s, in an $SU(2)_R$ invariant formulation or can be written in and $N = 1$ fashion directly in terms of X_{ij}, $\bar{X}_{ij}$ and their hermitean conjugates. In any case, it is clear that by giving digonal vev's to X 's ($\bar{X}$'s)

$$\langle X_{ij} \rangle = x_{ij} \mathbf{I}_n, \qquad \text{for all } i, j \tag{3.33}$$

compatible with the D-flatness conditions, we can break $SU(n)^N$ down to the diagonal $SU(n)$, with a massless spectrum coinciding with that of $N = 4$ SYM theory for $SU(n)$ gauge group. A similar flow, from $N = 1$ to $N = 4$, has been studied in [8] and [16] in the case of the ALE space $\mathbb{C}_3 / \mathbb{Z}_3$.

Notice that we can have intermediate possibilities for the IR point. In terms of the geometry, this can happen when some of the ALE centers x_i coincide with each other. Recalling the ALE metric, we have already seen that in the UV

$$V \sim \frac{N}{|\vec{x}|}, \qquad |\vec{x}| \to \infty \,.$$

In the IR, if we let M centers, $M < N$ to coincide with $\vec{x}_1$ say, and zoom near $\vec{x}_1$, we have

$$V \sim \frac{M}{|\vec{x} - \vec{x}_1|}, \qquad \vec{x} \to \vec{x}_1 \,. \tag{3.34}$$

The ALE geometry then develops a $\mathbb{Z}_M$ singularity in the IR. Therefore, all possibilities with any values of N and M should be allowed as long as $M < N$. We can also compute the ratio of the central charges by repeating the same procedure as in the previous section and end up with the result

$$\frac{a_{IR}}{a_{UV}} = \frac{c_{IR}}{c_{UV}} = \frac{M}{N} < 1\,. \tag{3.35}$$

On the other hand, on the field theory side, we can partially Higgs the gauge group $SU(n)^N$ down to $SU(n)^M$, for any $M < N$. That is we have flows between the corresponding quiver diagrams.

3.2 RG flows in Type I$'$ string theory

As we mentioned, type I$'$ is obtained from type I theory by two T-duality transformations along the two cycles of T^2. In this process, D9-branes will become D7-branes and the $SO(32)$ gauge group is broken to $SO(8)^4$, corresponding to the four fixed points of T^2. It has been shown in [15], that the resulting theory is dual to type IIB theory on $T^2/(-1)^{F_L}\Omega I_2$. One then considers a stack of D3-branes near one of the fixed points and in the near horizon geometry one gets $AdS_5 \times S^5/\mathbb{Z}_2$. This corresponds to a dual $N = 2$ CFT, with $USp(2n)$ gauge group and $SO(8)$ global flavor symmetry [33, 34], with matter hypermultiplets in the antisymmetric representation of $USp(2n)$ and also in the (real) $(2n, 8)$ of $USp(2n) \times SO(8)$. In our case, we are replacing $\mathbb{R}^4$ with ALE space, or in the orbifold limit, with $\mathbb{R}^4/\mathbb{Z}_N$. Similar to the type IIB case, the UV field theory will be obtained by performing the orbifold $\mathbb{Z}_N$ projection of the above field content, which in turn will be recovered at the IR point after Higgsing.

On the supergravity side, we will restrict our analysis to the two-derivative terms in the affective action. Therefore, the ansatz of the previous subsection can be carried over to this case. In particular, the Bianchi identity for the 5-form will be unchanged. Otherwise, one would have to switch on also D7-brane instantons on the ALE space in order to compensate for the $R \wedge R$ term present on the right-hand side of the Bianchi identity at order $\mathcal{O}(\alpha')$. The analysis in this case is closely similar to the previous case apart from the facts that we start with 16 supercharges in ten dimensions rather than 32, and the final equation for e^{-4f} is the same as before. Following similar analysis as in the previous subsection, we can show that the solution interpolates between $AdS_5 \times S^5/(\mathbb{Z}_N \times \mathbb{Z}_2)$ in the UV and $AdS_5 \times S^5/\mathbb{Z}_2$ in the IR, with $\mathbb{Z}_2$ being the orientation reversal operator Ω.

As mentioned above, the field theory interpretation will involve flows between $N = 2$ quiver gauge theories with different gauge groups. The gauge group in the UV will be obtained by considering orbifolding/orientifolding a system of D3/D7 branes,

whereas the IR group will be obtained by Higgsing, like in type IIB case. According to [17], the choice of gauge groups depends on the values of N as well as on the choice of a $\mathbb{Z}_N$ phase relating Ω and $\mathbb{Z}_N$ projections. In our case, the D3-brane worldvolume gauge theory descends from the theory on D5-brane for which $\gamma(\Omega)^t = -\gamma(\Omega)$. In what follows, we will use the notations of [17] and also refer the reader to this reference for more detail on the quiver gauge theory. We first review the consistency conditions for the $\mathbb{Z}_N \times \mathbb{Z}_2$ actions [17]

$$
\begin{aligned}
\Omega^2 = 1 : & \quad \gamma(\Omega) = \chi(\Omega)\gamma(\Omega)^t, \\
\Omega g = g\Omega : & \quad \gamma(g)\gamma(\Omega)\gamma(g)^t = \chi(g,\Omega)\gamma(\Omega), \\
g^n = 1 : & \quad \gamma(g)^n = \chi(g)1
\end{aligned}
\tag{3.36}
$$

where $g \in \mathbb{Z}_N$ and $\chi(\Omega)$, $\chi(g)$ and $\chi(g,\Omega)$ are phases. As shown in [17], we can set $\chi(g) = 1$. Furthermore, we are interested in the case of $\chi(\Omega) = -1$ on the D3-branes and $\chi(\Omega) = +1$ on the D7's. In type I theory, there are five cases to consider, but only three of them are relevant for us. These are N odd, $\chi(g,\Omega) = 1$, N even, $\chi(g,\Omega) = 1$ and N even, $\chi(g,\Omega) = \xi$ with $\xi = e^{\frac{2\pi i}{N}}$. We now consider RG flows in these cases.

3.2.1 $\chi(g,\Omega) = 1$, N odd

In this case, the gauge group is given by

$$
\begin{aligned}
G_1 &= USp(v_0) \times [U(v_1) \times U(v_2) \times \ldots \times U(v_{\frac{N-1}{2}})] \\
&= \{U_0, U_1, \ldots, U_{N-1} | U_i U_{N-i}^t = 1, 1 \leq i \leq N-1\}.
\end{aligned}
\tag{3.37}
$$

Our convention is that $USp(2n)$ has rank n. The full configuration involves also the quiver theory on D9-branes which give rise to D7-branes in our case. Our main aim here is to study the symmetry breaking of the gauge group on D3-branes. The presence of D7-branes is necessary to make the whole system conformal. For the UV quiver gauge theory to be conformal, we choose $v_0 = v_1 = \ldots = v_{\frac{N-1}{2}} = n$ with an appropriate number of D7-branes such that the field theory beta function vanishes. Using the notation of [17], we denote the vector spaces associated to the nodes of the inner quiver, the D3-branes, by V_i and those of the outer quiver on D7-branes by W_i. There is also an identification of the nodes $V_i = V_{N-i}$ and similarly for W_i's, see [17]. This condition gives rise to the relation between the gauge groups of different nodes as shown in (3.37).

The gauge theory on the D7-branes is described by similar gauge group structure but with $USp(2n)$ replaced by $O(2n)$ due to the opposite sign of $\chi(\Omega)$. In addition to the vector multiplets, there are hypermultiplets, X, $\bar{X}$, associated to the links connecting the V_i's and I, J related to the links connecting V_i's and W_i's, in bifundamental

representations of the respective gauge groups. The vanishing of the beta function can be achieved by setting $w_0 = 4$, $w_{\frac{N-1}{2}} = w_{\frac{N+1}{2}} = 2$ and the other w_i's zero. Notice that the non trivial gauge groups on the outer quiver are associated with two types of the nodes of the inner quiver. The first type consists of the nodes with USp gauge groups while the second type contains nodes connected to each other by antisymmetric scalars. The corresponding outer gauge groups for these two types are $SO(4)$ and $U(2)$, respectively.

It is easy to see the reason for this pattern of inner/outer gauge groups: the point is that for the inner nodes with $U(2n)$ gauge groups and connected by $U(2n)$ bifundamental scalars, the corresponding part of the quiver diagram is essentially the same as the quiver diagram arising from type IIB theory in which all the gauge groups are unitary. It is well-known that this quiver gauge theory is supeconformal without any extra field contents.

As observed in [17, 35], the above construction matches with the ADHM construction of $SO(n)$ instantons on ALE spaces: for example, the assignement of D7-brane gauge group given above means that, at the boundary of the ALE space, which has fundamental group $\pi_1 = \mathbb{Z}_N$, the $SO(8)$ flat connection has holonomy which breaks $SO(8)$ down to $SO(4) \times U(2) \sim SO(4) \times SU(2) \times U(1)$. On the other hand, G_1 is the ADHM gauge group, related to the number of instantons (D3-branes).

We now consider a Higgsing of this theory, and we need to be more precise about the representations of the matter fields. The nodes are connected to each other by the bifundamental scalars X and $\bar{X}$. These scalars are subject to some constraints given by

$$
\begin{aligned}
X_{01} &= -(X_{N-1,0}\omega_{2n})^t, & X_{\frac{N-1}{2},\frac{N+1}{2}} &= -(X_{\frac{N-1}{2},\frac{N+1}{2}})^t, \\
X_{i,i+1} &= (X_{N-i-1,N-i})^t, & 1 \leq i &\leq \frac{N-3}{2}, \\
\bar{X}_{10} &= (\omega_{2n}\bar{X}_{0,N-1})^t, & \bar{X}_{\frac{N+1}{2},\frac{N-1}{2}} &= -(\bar{X}_{\frac{N+1}{2},\frac{N-1}{2}})^t, \\
\bar{X}_{i+1,i} &= (\bar{X}_{N-i,N-i-1})^t, & 1 \leq i &\leq \frac{N-3}{2}
\end{aligned}
\tag{3.38}
$$

where ω_{2n} represents the symplectic form of dimension $2n$. We will show that after the RG flow, the theory will flow to $USp(2n)$, $N = 2$ gauge theory in which the gauge group $USp(2n)$ is the diagonal subgroup of the $USp(2n)$ and $USp(2n)$ subgroups of all the $U(2n)$'s. We first illustrate this with the simple case of $N = 5$. The corresponding quiver diagram is shown in Figure 1. In the figure, the outer quiver and the inner one are connected to each other by scalar fields I_i, J_i (we have omitted the $\bar{X}$'s on the diagram). Notice that the gauge groups in the outer quiver are orthogonal and unitary groups due to the opposite sign of $\chi(\Omega)$. We will be interested in the Higgs

branch, i.e. we set the vev's of the scalars in the vector multiplets to zero. Furthermore, we will also set $\langle I \rangle = \langle J \rangle = 0$ in all the cases we will discuss in the following. The D-flatness conditions are then obtained from those of the type IIB case by suitable projections/identifications on the X's and $\bar{X}$'s. The main difference, compared to the type IIB case, comes from the gauge group, which involve an $USp(2n)$ factor at the 0-th vertex and has $U(2n)$ factors which are related in the way indicated in Figure 1: as a result, the corresponding FI terms obey $\vec{D}_0 = 0$, $\vec{D}_1 = -\vec{D}_4$, $\vec{D}_2 = -\vec{D}_3$, with similar relations for higher odd N.
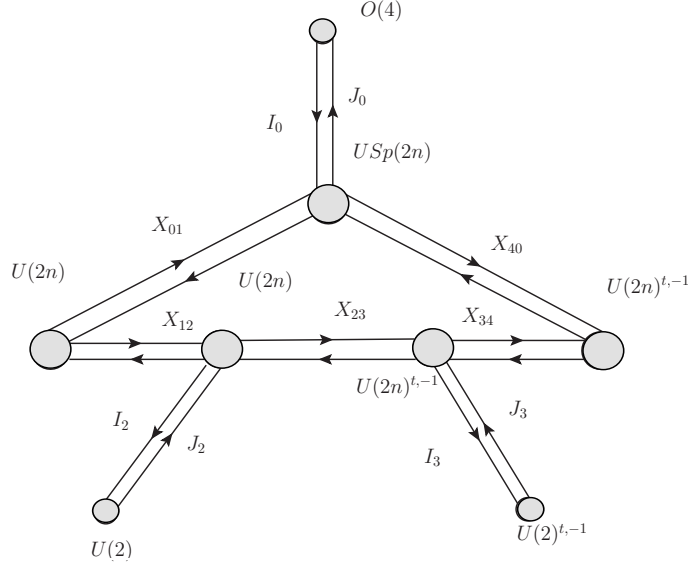

Figure 1: Quiver diagram for $\chi(\Omega) = -1$, $\chi(g, \Omega) = 1$ and $N = 5$.

We will give only the flows in which X and $\bar{X}$ acquire vev's. The above conditions then give

$$X_{01} = -X_{40}^t, \qquad X_{12} = X_{34}^t, \qquad X_{23} = -X_{23}^t \qquad (3.39)$$

and similarly for $\bar{X}$. We choose the vev's as follows

$$\langle X_{01} \rangle = a\mathbf{I}_{2n}, \qquad \langle X_{12} \rangle = b\mathbf{I}_{2n}, \qquad \langle X_{23} \rangle = c\omega_{2n} \qquad (3.40)$$

where a, b and c are constants. The vev's for $\bar{X}$ are similar but with different parameters $\bar{a}$, $\bar{b}$ and $\bar{c}$. Notice also that we only need to give vev's to the independent fields since the vev's of other fields can be obtained from (3.39). From now on, we will explicitly analyze only the X's. The analysis for $\bar{X}$'s follows immediately.

The field X_{ij} transforms as $g_i X_{ij} g_j^{-1}$ where g_i and g_j are elements of the two gauge groups, G_i and G_j, connected by X_{ij}. The unbroken gauge group is the subgroup of

$USp(2n) \times U(2n) \times \ldots \times U(2n)$ that leaves all these vev's invariant. The invariance of X_{01} requires that g_1 is a symplectic subgroup of $U(2n)$ and $g_1 = g_0$. The invariance of X_{12} imposes the condition $g_1 = g_2 = g_0$ and so on. In the end, we find that the gauge group in the IR is $USp(2n)_{\text{diag}}$. For any odd N, the whole process works in the same way apart from the fact that there are more nodes similar to X_{12}. These nodes can be given vev's proportional to the identity. Taking this into account, we end up with scalar vev's

$$\langle X_{01} \rangle = a_{01} \mathbf{I}_{2n}, \qquad \langle X_{\frac{N-1}{2}, \frac{N+1}{2}} \rangle = a_{\frac{N-1}{2}, \frac{N+1}{2}} \omega_{2n},$$

$$\langle X_{i,i+1} \rangle = a_{i,i+1} \mathbf{I}_{2n}, \qquad 1 \leq i \leq \frac{N-3}{2}, \tag{3.41}$$

and the unbroken gauge group is $USp(2n)_{\text{diag}}$. In addition, one can verify that the masless spectrum is precisely that of the superconformal $USp(2n)$ theory with $SO(8)$ global symmetry described at the beginning of this section.

3.2.2 $\chi(g, \Omega) = 1$, N even

In this case, we have the gauge group

$$G_2 = USp(v_0) \times [U(v_1) \times \ldots \times U(v_{\frac{N}{2}-1})] \times USp(v_{\frac{N}{2}})$$

$$= \left\{ U_0, \ldots, U_{N-1} | U_i U_{N-i}^t = 1, 1 \leq i \leq N-1, i \neq \frac{N}{2} \right\}. \tag{3.42}$$

Compared to the previous case, there is an additional $USp(v_{\frac{N}{2}})$ gauge group at the $\frac{N}{2}^{\text{th}}$ node. As before, we choose $v_0 = v_1 = \ldots = v_{\frac{N}{2}} = 2n$ and $w_0 = w_{\frac{N}{2}} = 4$ with other w_i's being zero, corresponding to the breaking of the D7-brane gauge group from $SO(8)$ down to $SO(4) \times SO(4)$. The scalars are subject to the constraints

$$X_{01} = \omega_{2n}(X_{N-1,o})^t, \qquad X_{\frac{N}{2}, \frac{N+2}{2}} = -\omega_{2n}(X_{\frac{N-2}{2}, \frac{N}{2}})^t,$$

$$X_{i,i+1} = (X_{N-i-1,N-i})^t, \qquad 1 \leq i < \frac{N-2}{2}. \tag{3.43}$$

The corresponding quiver diagram for $N = 4$ is shown in Figure 2.

As for the FI terms in this case, clearly $\vec{D}_0 = \vec{D}_2 = 0$ and $\vec{D}_1 = -\vec{D}_3$, with the obvious generalization for higher even N. We can choose the following vev's to Higgs the theory

$$\langle X_{01} \rangle = x_{01} \mathbf{I}_{2n}, \qquad \langle X_{\frac{N-2}{2}, \frac{N}{2}} \rangle = x_{\frac{N-2}{2}, \frac{N}{2}} \mathbf{I}_{2n},$$

$$\langle X_{i,i+1} \rangle = x_{i,i+1} \mathbf{I}_{2n}, \qquad 1 \leq i < \frac{N-2}{2}. \tag{3.44}$$
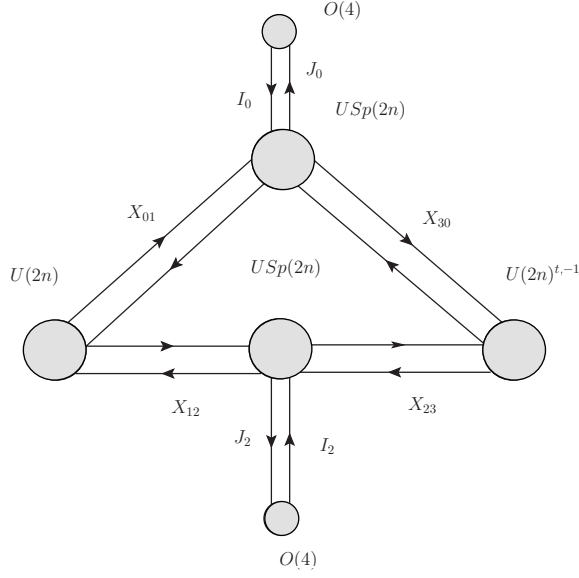
Figure 2: Quiver diagram for $\chi(\Omega) = -1$, $\chi(g, \Omega) = 1$ and $N = 4$.

The symmetry breaking is the same as in the previous case. These vev's are invariant under the unbroken gauge group $USp(2n)_{\text{diag}}$, and one can verify that massless hypermultiplets fill the spectrum of the $N = 2$ theory discussed in the previous case.

3.2.3 $\chi(g, \Omega) = \xi$, N even

It is possible to choose $\chi(g, \Omega) = \xi$ for N even as shown in [17], and this is our last case. We adopt the range of the index i from 1 to N in this case. The relevant gauge group is given by

$$
\begin{aligned}
G_3 &= U(v_1) \times U(v_2) \times \ldots \times U(v_{\frac{N}{2}}) \\
&= \{U_1, \ldots, U_N | U_i U_{N-i+1}^t = 1, 1 \le i \le N\}.
\end{aligned}
\tag{3.45}
$$

We are interested in the case $v_1 = v_2 = \ldots = v_{\frac{N}{2}} = 2n$ and $w_1 = w_{\frac{N}{2}} = 2$ with other w_i's being zero, i.e. the D7 gauge group is now broken down to $U(2) \times U(2)$. The conditions on the scalar fields are

$$
\begin{aligned}
X_{N1} &= -X_{N1}^t, \qquad X_{\frac{N}{2}, \frac{N+2}{2}} = -(X_{\frac{N}{2}, \frac{N+2}{2}})^t, \\
X_{i,i+1} &= (X_{N-i,N-i+1})^t, \qquad 1 \le i \le \frac{N-2}{2}.
\end{aligned}
\tag{3.46}
$$

The quiver diagram for $N = 4$ and $\xi = i$ is shown in Figure 3. Notice the relations $\vec{D}_1 = -\vec{D}_4$, $\vec{D}_2 = -\vec{D}_3$ and so on for higher even N. There are two possibilities for

Higgsing this theory. The first one involves only the vev's

$$\langle X_{i,i+1}\rangle = b_{i,i+1}\mathbf{I}_{2n}, \qquad 1 \le i \le \frac{N-2}{2}. \tag{3.47}$$

The unbroken gauge group is the diagonal subgroup of $U(v_1) \times \ldots \times U(v_{\frac{N}{2}})$, $U(2n)_{\text{diag}}$. The second possibility is to give vev's to all scalars including the antisymmetric ones

$$\langle X_{i,i+1}\rangle = b_{i,i+1}\mathbf{I}_{2n}, \qquad 1 \le i \le \frac{N-2}{2},$$
$$\langle X_{N1}\rangle = b_{N1}\omega_{2n}, \qquad \langle X_{\frac{N}{2},\frac{N+2}{2}}\rangle = b_{\frac{N}{2},\frac{N+2}{2}}\omega_{2n}. \tag{3.48}$$

In this case, the resulting gauge group is further broken down to $USp(2n)_{\text{diag}}$.
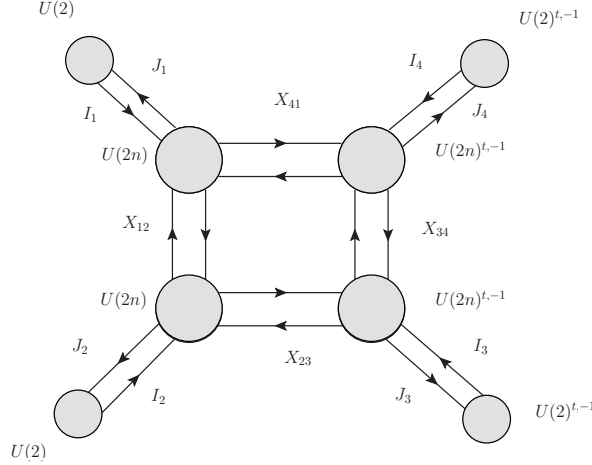

Figure 3: Quiver diagram for $\chi(\Omega) = -1$, $\chi(g,\Omega) = i$ and $N = 4$.

4. Symmetry breaking and geometric interpretations

In this section, like in the type IIB case, we consider more general symmetry breaking patterns in the field theory and match them with the possible flows emerging from the supergravity solution. This involves the cases in which the gauge groups in the quiver gauge theory are not completely broken down to a single diagonal subgroup. After symmetry breaking, the IR CFT is again a quiver gauge theory with a reduced number of gauge groups, and of course, the number of nodes is smaller. We will show that some symmetry breaking patterns are not possible on the field theory side, at least by giving simple vev's to scalar fields.

We now consider the possibility of RG flows from a UV CFT which is a quiver gauge

theory with the corresponding geometry $AdS_5 \times S^5/(\mathbb{Z}_N \times \mathbb{Z}_2)$ to an IR CFT which is associated to the geometry $AdS_5 \times S^5/(\mathbb{Z}_M \times \mathbb{Z}_2)$ and $M < N$. We saw that in the type IIB case this was always possible, and geometrically it was related to geometries developing a $\mathbb{Z}_M$ orbifold singularity obtained by bringing M centers together in the smooth ALE metric.

Let us start from the field theory side. It is easy to see that it is not always possible to have a flow from one quiver diagram to the other. For example, we consider a flow from the diagram in Figure 1, $N = 5$, to Figure 2, $N = 4$. This can be done by giving a vev to X_{23} and $\bar{X}_{32}$ which transform in the antisymmetric tensor representation of $U(2n)$. The gauge group $U(2n)$ at the node v_2 and v_3 will be broken to $USp(2n)$. The resulting IR theory is then described by the quiver diagram in Figure 2. Continuing the process by Higgsing Figure 2 to the diagram with $N = 3$, we find that it is not possible to completely break the $USp(2n)$ gauge group at v_2 with the remaining scalars transforming in the antisymmetric tensor representation of $U(2n)$. It might be achieved by giving a vev to complicated composite operators, but we have not found any of these operators. Note also that this is the case only for reducing the value of N by one unit. If we Higgs the $N = 6$ to $N = 4$ or in general N to $N - 2$, this flow can always be achieved by giving vev's to X_{12} and $\bar{X}_{21}$. The gauge groups $U(2n)$ at v_1 and v_2 as well as at their images v_{N-1} and v_{N-2} will be broken to $U(2n)_{\text{diag}}$. The resulting quiver diagram is the same type as the original one with two nodes lower. What we are interested in is the problematic cases in which the flow connects two types of diagrams and lowers N by one unit.

We now begin with a diagram of the type shown in Figure 3. As mentioned in the previous section, this type is only possible for even N. It is easily seen that giving a vev to the $U(2n)$ antisymmetric scalars X_{1N} and $\bar{X}_{N1}$ reduces the diagram to the $N-1$ diagram of the type shown in Figure 1. Furthermore, a diagram with $N-2$ nodes of the type in Figure 2 can be obtained by giving an additional vev to $X_{\frac{N}{2}, \frac{N+2}{2}}$ and $\bar{X}_{\frac{N+2}{2}, \frac{N}{2}}$. Now, the problem arises in deriving this diagram from the odd N diagram. As before, the $USp(2n)$ gauge groups at v_0 must be completely broken leaving only scalars in the $U(2n)$ antisymmetric tensor. Actually, it seems to be impossible to obtain this type of quiver diagrams from any of the other two types by Higgsing in a single or multiple steps since the process involves the disappearance of the USp gauge group.

We now discuss how the above field theory facts match with the geometry on the supergravity side. We will follow the approach in [18], where some peculiarities of type I string theory on $\mathbb{Z}_N$ orientifolds where clarified. The idea is to use a regular D1-brane (in type I theory), to probe the background geometry, following the same logic explained for the type IIB case in the previous subsection. In that case, we saw that one could reproduce the full smooth ALE geometry by switching on FI terms, which

are background values of closed string moduli. The $\mathbb{Z}_N$ projection has generically the effect of reducing unitary groups down to SO/USp subgroups and/or of identifying pairs of unitary groups, in a way which depends on the details of the projection. We can indeed consider a probe D1-brane in the present orientifold context and derive its effective field theory for the three cases discussed in the previous section by assuming an orthogonal projection $\chi(\Omega) = 1$. In the following, the diagrams in Figures 1, 2 and 3 will be referred to as type I, II and III quivers, respectively.

For the case 1, $\chi(g, \Omega) = 1$, $N = 2m + 1$ odd, we will have m pairs of conjugate $U(1)$'s as gauge groups, (with an $O(1)$ "gauge group" at the 0-th vertex of the inner quiver diagram of Figure 1), with appropriate identifications of the scalars X, $\bar{X}$. For example for $N = 3$, we have $X_{01} = X_{20}$ plus X_{12}, similarly for $\bar{X}$ fields. Consequently, for the FI terms, we will have $D_0 = 0$ and $D_i = -D_{N-i}$, $i = 1, \ldots, m$. Translating these data to the ALE centers $\vec{x}_i$ via (3.32) as in the previous subsection, we see that for $N = 2m + 1$ there are m $\mathbb{Z}_2$ singularities. There is in addition a simple pole in the function V, which is however a smooth point in the geometry as long as it is kept distinct from the other poles. The function V in the ALE metric (2.10) is then given by

$$V = \frac{1}{|\vec{x} - \vec{x}_1|} + \sum_{i=2}^{m+1} \frac{2}{|\vec{x} - \vec{x}_i|} \, . \tag{4.1}$$

If we choose the IR point by setting $\vec{x} \to \vec{x}_1$, we end up with the flow from $N = 2$ quiver gauge theory of type I to the $N = 2$, $USp(2n)_{\text{diag}}$ gauge theory. The flow from type I quiver with $N = 2m + 1$ to type I quiver with $N = 2m - 1$ can be obtained by choosing $\vec{x} \to \vec{x}_1$ with $\vec{x}_i = \vec{x}_1$ for $i = 2, \ldots, m - 1$. Finally, the flow to type II quiver in the case 2 can be achieved by setting $\vec{x}_i = \vec{x}_2$ for $i = 3, \ldots, m - 1$ and $\vec{x} \to \vec{x}_2$.

For the case 2, $\chi(g, \Omega) = 1$, $N = 2m$ even, we will have $O(1)$ at the nodes 0 and m, and the remaining $U(1)$'s are pairwise conjugate, and there are obvious identifications for the X and $\bar{X}$ fields. Consequently, $\vec{D}_0 = \vec{D}_m = 0$ and $\vec{D}_i = -\vec{D}_{N-i}$, $i = 1, \ldots, m-1$. In terms of the ALE metric, we see that there are m $\mathbb{Z}_2$ singularities. The corresponding V function is

$$V = \sum_{i=1}^{m} \frac{2}{|\vec{x} - \vec{x}_i|} \, . \tag{4.2}$$

The possible flows are the following. First of all, to obtain the $N = 2$, $USp(2n)_{\text{diag}}$ gauge theory in the IR, we choose $\vec{x} \to \vec{x}'$ where $\vec{x}'$ is any regular point. The full Green function $G(x, x')$ will behave in the same way as $\vec{x} \sim \vec{x}_i$. In this case, the IR geometry is a smooth space. Another possible flows are given by Higgsing type II diagram with $N = 2m$ to the same type with $N = 2m - 2$. This is achieved by setting $\vec{x}_i = \vec{x}_1$ for $i = 2, \ldots, m - 1$ and $\vec{x} \to \vec{x}_1$.

Finally, for the case 3, $\chi(g,\Omega) = \xi$, therefore $N = 2m$ even, we have m pairs of conjugate $U(1)$ factors and consequently $\vec{D}_i = -\vec{D}_{N-1-i}$, $i = 0, \ldots, m$, which implies $m - 1$ $\mathbb{Z}_2$ singularities plus two smooth points in the geometry. The function V is given by

$$V = \frac{1}{|\vec{x} - \vec{x_1}|} + \frac{1}{|\vec{x} - \vec{x}_{2m}|} + \sum_{i=2}^{m} \frac{2}{|\vec{x} - \vec{x}_i|}. \tag{4.3}$$

The flow from type III quiver to $N = 2$, $USp(2n)_{\text{diag}}$ gauge theory is given by $\vec{x} \to \vec{x}_1$ or $\vec{x} \to \vec{x}_{2m}$. If we choose $\vec{x} \to \vec{x}_1$ and $\vec{x}_i = \vec{x}_1$ for $i = 2, \ldots, 2m-1$, we obtain the flow from type III quiver with $N = 2m$ to type I quiver with $N = 2m - 1$. On the other hand, if we choose $\vec{x} \to \vec{x}_2$ and $\vec{x}_i = \vec{x}_2$ for $i = 3, \ldots, 2m-1$, we find a flow from type III quiver with $N = 2m$ to type II quiver with $N = 2m - 2$. The flow from type III quiver with $N = 2m$ to type III quiver with $N = 2m - 2$ is given by setting $\vec{x} \to \vec{x}_1$ and $\vec{x}_i = \vec{x}_{2m} = \vec{x}_1$ for $i = 2, \ldots, 2m - 2$.

Notice that the V in (4.3) cannot be obtained from either (4.1) or (4.2) since both of them have none or only one single singularities while V in (4.3) has two. Furthermore, the flow from type II quiver to type I quiver is not allowed because there is no single singularity in (4.2), but there is one in (4.1). All the flows given above exactly agree with those obtained from the field theory side. So, we see that the effect of the Ω projection is to remove some of the blowing up, closed string, moduli and therefore the geometry cannot be completely smoothed out. Generically there remain $\mathbb{Z}_2$ singularities. Of course higher singularities can be obtained by bringing together the centers surviving the Ω projection. We summarize all possible flows in table 4. The UV geometry is always $AdS_5 \times S^5/(\mathbb{Z}_N \times \mathbb{Z}_2)$ with $\vec{x}_{\text{UV}} \to \infty$. The V_{UV} is given by that of (2.10) while V_{IR}'s can be obtained by the $\vec{x}_{\text{IR}}$ given in the table via (4.1), (4.2) and (4.3). In the Flow column, the notation I$(2m+1)\to$ II$(2m)$ means the flow from type I quiver with $N = 2m+1$ to type II quiver with $N = 2m$ etc. The $N = 2$, $USp(2n)_{\text{diag}}$ gauge theory is denoted by I(1). The ALE centers are labeled in the same ordering as in equations (4.1), (4.2) and (4.3). Finally, $\vec{x}_{\text{IR}}$'s are the IR points with the notation $\vec{x}'$ denoting any regular point away from the ALE center $\vec{x}_i$'s.

5. Conclusions

We have studied RG flow solutions in the four and two dimensional field theories on the background of the A_N ALE space. The flows in two dimensions are similar to the solution given in [10] with the flat four dimensional space replaced by the ALE space. The flows are vev flows driven by a vacuum expectation value of a marginal operator as in the solutions of [10]. The dual field theory description is that of the (2,0) UV CFT

Flow	$\vec{x}_{\text{IR}}$
$\text{I}(2m+1) \to \text{I}(1)$	$\vec{x}_1$
$\text{II}(2m) \to \text{I}(1)$	$\vec{x}'$
$\text{III}(2m) \to \text{I}(1)$	$\vec{x}_1$
$\text{I}(2m+1) \to \text{I}(2n+1, n < m)$	$\vec{x}_i = \vec{x}_1,\ i = 2, \ldots, n$
$\text{I}(2m+1) \to \text{II}(2n, n \leq m)$	$\vec{x}_i = \vec{x}_2,\ i = 3, \ldots, n+1$
$\text{II}(2m) \to \text{II}(2n, n < m)$	$\vec{x}_i = \vec{x}_1,\ i = 2, \ldots, n$
$\text{III}(2m) \to \text{III}(2n, n < m)$	$\vec{x}_i = \vec{x}_1 = \vec{x}_{2m},\ i = 2, \ldots, n$
$\text{III}(2m) \to \text{II}(2n, n \leq m-1)$	$\vec{x}_i = \vec{x}_2,\ i = 2, \ldots, n$
$\text{III}(2m) \to \text{I}(2n+1, n \leq m-1)$	$\vec{x}_i = \vec{x}_1,\ i = 2, \ldots, n$

Table 1: All possible RG flows of the $N = 2$ quiver gauge theories arising in type I$'$ theory.

flows to the (4,0) theory in the IR. The corresponding geometries are $AdS_3 \times S^3/\mathbb{Z}_N$ and $AdS_3 \times S^3$. We have computed the central charges in both the UV and IR to curvature squared terms in the bulk. The ratio of the central charges to the leading order contains a factor of N as expected from the ratio of the volumes of the S^3 and $S^3/\mathbb{Z}_N$ on which the six dimensional supergravity is reduced.

In type IIB theory, we have studied a flow solution describing an RG flow in four dimensional field theory. It involves the Green's function on $\text{ALE} \times \mathbb{R}^2$, which we were unable to find explicitly, but whose existence is guaranteed. The solution interpolates between $AdS_5 \times S^5/\mathbb{Z}_N$ and $AdS_5 \times S^5$. The flow is again a vev flow driven by a vacuum expectation value of a relevant operator of dimension two. The flow drives the $N = 2$ quiver gauge theory with the gauge group $SU(n)^N$ in the UV to the $N = 4$ $SU(n)_{\text{diag}}$ supersymmetric Yang-Mills theory in the IR. The hypermultiplets acquire vacuum expectation values proportional to the identity matrix and break $SU(n)^N$ to its diagonal subgroup $SU(n)_{\text{diag}}$ in the IR. The central charges a and c have also been computed to the curvature squared terms.

Moreover, we have studied a flow solution in type I$'$ theory. The flow solution interpolates between $AdS_5 \times S^5/(\mathbb{Z}_N \times \mathbb{Z}_2)$ and $AdS_5 \times S^5/\mathbb{Z}_2$ where the $\mathbb{Z}_2$ is $(-1)^{F_L}\Omega I_2$. The flow is again driven by a vacuum expectation value of a relevant operator of dimension two. In contrast to the type IIB case, the field theory description is more complicated and more interesting. There are three cases to be considered. For N odd and $\chi(g, \Omega) = 1$, the flow drives the $N = 2$ quiver gauge theory with the gauge group $USp(2n) \times U(2n) \times \ldots \times U(2n)$ to the $N = 2$, $USp(2n)_{\text{diag}}$ gauge theory. For N even and $\chi(g, \Omega) = 1$, the flow describes an RG flow from $N = 2$ quiver $USp(2n) \times U(2n) \times \ldots \times U(2n) \times USp(2n)$ gauge theory to $N = 2$, $USp(2n)_{\text{diag}}$ gauge theory. Finally, for N even and $\chi(g, \Omega) = e^{\frac{2\pi i}{N}}$, we find the flow from $N = 2$ quiver $U(2n) \times \ldots \times U(2n)$

gauge theory to $N = 2$, $U(2n)_{\mathrm{diag}}$ gauge theory for vanishing expectation values of the antisymmetric bifundamental scalars. With non-zero antisymmetric scalar expectation values, the gauge group in the IR is reduced to $USp(2n)_{\mathrm{diag}}$.

We have also generalized the previous discussion to RG flows between two $N = 2$ quiver gauge theories in both type IIB and type I$'$ theories. The gravity solution interpolates between $AdS_5 \times S^5/(\mathbb{Z}_N \times \mathbb{Z}_2)$ and $AdS_5 \times S^5/(\mathbb{Z}_M \times \mathbb{Z}_2)$ geometries. In type IIB theory, the flows work properly as expected from the field theory side in a strightforward way. In type I$'$ theory, field theory considerations forbid some symmetry breaking patterns. However, this is in agreement with the geometrical picture, after one takes into account the restrictions put on the geometry by the orientifolding procedure.

We conclude this paper with a few comments regarding the type I$'$ case. If we include higher order terms in the effective action, we need, among other things, to switch on the $F \wedge F$ to ensure the Bianchi identity for the 5-form

$$d\tilde{F}^{(5)} = \frac{\alpha'}{4}(\mathrm{Tr}R \wedge R - \mathrm{Tr}F \wedge F)\delta^{(2)}(\vec{z}) \tag{5.1}$$

F being the field strength of the $SO(8)$ gauge group and $\vec{z}$ a coordinate on the transverse $\mathbb{R}^2$. In particular, we need to include $SO(8)$ instantons on the ALE spaces (with the standard metric, the warp factor being irrelevant due to conformal invariance). It would be interesting to relate ALE's instanton configurations to the pattern of symmetry breaking of the global $SO(8)$ group involved in the various flows discussed in the previous Section. As already mentioned, the UV group is determined by the holonomy of the flat connection at the ALE's boundary, which is in turn part of the ADHM data. It would be interesting to understand in a similar way the IR group.

Acknowledgments

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